Constraint structure of O(3) nonlinear sigma model revisited

Constraint structure of O(3) nonlinear sigma model revisited

Soon-Tae Hong[1]**, Yong-Wan Kim**[1]**, Young-Jai Park**[1] **and K D Rothe**[2]

[1] Department of Physics and Basic Science Research Institute, Sogang University,
CPO Box 1142, Seoul 100-611, Korea
[2] Institut für Theoretische Physik, Universität Heidelberg, Philosophenweg 16,
D-69120 Heidelberg, Germany

Abstract
We study the constraint structure of the O(3) nonlinear sigma model (NLSM) in the framework of the Lagrangian, symplectic, Hamilton–Jacobi as well as the Batalin–Fradkin–Tyutin embedding procedure.

PACS numbers: 45.20.Jj, 02.40.Re

1. Introduction

Since the $(2 + 1)$-dimensional O(3) nonlinear sigma model (NLSM) was first discussed [1, 2], there have been many attempts to improve this soliton model associated with the homotopy group $\pi_2(S^2) = Z$. Moreover, the O(3) NLSM with the Hopf term [3], as well as the CP^1 model associated with the O(3) NLSM via the Hopf map projection from S^3 to S^2, were canonically quantized [4–6] in the standard Dirac formalism [7]. Furthermore, the geometrical constraints involved in the soliton models such as the Skyrmion models [8, 9], O(3) NLSM [10] and CP^1 model [11] have recently been analysed in the framework of the Batalin–Fradkin–Tyutin (BFT) scheme [12–14], to obtain the corresponding first-class Hamiltonians and BRST [15] invariant effective Lagrangians [16, 17].

On the other hand, an alternative symplectic scheme [18] has been proposed for treating first-order Lagrangians. This scheme has been applied to various models [19–22] and has recently been extended to include BRST invariant embeddings [23]. Moreover, since the Hamilton–Jacobi (HJ) scheme [24] was first applied to constrained systems [25], there have been further developments in this direction to quantize singular systems [26–28]. In particular, the HJ description of second-class constrained systems, which posed some problems in previous work [29], is shown to be complete after the addition of suitable integrability conditions, corresponding to the Dirac consistency conditions, requiring time independence of the constraints [28]. Recently, the O(3) NLSM was also considered in an extended HJ scheme [27] based on the extension of the second-class fields to the first-class fields previously

introduced in [10]. However, this extension of the HJ scheme continues to involve some subtleties still to be resolved.

In this paper, we will revisit the O(3) NLSM and examine the relation among the Lagrangian, symplectic, HJ and the BFT embedding scheme for this model. In section 2 we will analyse the constraint structure of this model from the point of view of the Lagrangian and symplectic algorithm. In section 3, we will investigate the HJ quantization of the O(3) NLSM taking account of the integrability conditions. In section 4 we then discuss the BFT quantization scheme for this model, and conclude in section 5.

2. NLSM in Lagrangian and symplectic schemes

In this section, we consider the Lagrangian and symplectic algorithms to generate the constraint structure of the O(3) NLSM, defined by the Lagrangian

$$\mathcal{L} = \frac{1}{2f}(\partial_\mu n^a)(\partial^\mu n^a) + n^0(n^a n^a - 1) \tag{2.1}$$

where n^a ($a = 1, 2, 3$) is a multiplet of three real scalar fields which parametrize an internal space S^2, and n^0 is a Lagrange multiplier field implementing the geometrical constraint $n^a n^a - 1 = 0$. From the Lagrangian (2.1) the canonical momenta conjugate to the field n^0 and the real scalar fields n^a are given by

$$\pi^0 = 0 \qquad \pi^a = \frac{1}{f}\partial_0 n^a \tag{2.2}$$

to yield the canonical Hamiltonian

$$\mathcal{H}^{(0)} = \frac{f}{2}\pi^a \pi^a + \frac{1}{2f}(\vec{\nabla} n^a)^2 - n^0(n^a n^a - 1). \tag{2.3}$$

Following [18], we rewrite the second-order Lagrangian (2.1) in first-ordered form as

$$\mathcal{L}^{(0)} = \pi^a \partial_0 n^a - \mathcal{H}^{(0)} \tag{2.4}$$

where $\mathcal{L}^{(0)}$ is to be treated as a function of the *Lagrangian* variables (n^0, n^a, π^a), and $\mathcal{H}^{(0)}$ is the canonical Hamiltonian (2.3). The original second-order formulation is recovered on shell, upon using the equation of motion for π^a. Note that since $\mathcal{H}^{(0)}$ depends on π^a and n^a, but not on their time derivatives, it can be regarded as the (level zero) symplectic potential. The Lagrangian is of the form

$$\mathcal{L}^{(0)} = a_\alpha \partial_0 \xi^\alpha - \mathcal{H}^{(0)} \tag{2.5}$$

where we have denoted the set of symplectic variables (n^a, π^a, n^0) collectively by ξ^α and the corresponding conjugate momenta by a_α. The dynamics of the model is then governed by the symplectic two-form matrix,

$$F_{\alpha\beta}^{(0)}(x, y) = \frac{\partial a_\beta(y)}{\partial \xi^\alpha(x)} - \frac{\partial a_\alpha(x)}{\partial \xi^\beta(y)} \tag{2.6}$$

via the equations of motion[3]

$$\int d^2 y \, F^{(0)}(x, y)\partial_0 \xi(y) = K^{(0)}(x) \tag{2.7}$$

---

[3] For notational convenience we will omit component indices for vectors and matrices except in cases where confusion may arise.

where

$$K_\alpha^{(0)}(x) = \frac{\delta}{\delta\xi^\alpha(x)} \int d^2y \, \mathcal{H}_0(y) = \begin{pmatrix} -2n^0(x)n^a(x) - \frac{1}{f}\vec{\nabla}^2 n^a(x) \\ f\pi^a(x) \\ -n^a(x)n^a(x) + 1 \end{pmatrix}. \quad (2.8)$$

In the O(3) NLSM the symplectic two-form matrix is given by

$$F^{(0)}(x, y) = \begin{pmatrix} O & -I & \vec{0} \\ I & O & \vec{0} \\ \vec{0}^T & \vec{0}^T & 0 \end{pmatrix} \delta^2(x - y) \quad (2.9)$$

where O and I stand for 3×3 null and identity matrices, respectively, and the superscript T denotes 'transpose'. The zero-level symplectic two-form has a zero mode[4] $v_y^{(0)}(x) = (\vec{0}, \vec{0}, 1)\delta^2(x - y)$, which generates the constraint

$$\int d^2y \, v_y^{(0)}(x)K^{(0)}(y) = -\Omega_1(x) = 0 \quad (2.10)$$

with

$$\Omega_1 = n^a n^a - 1. \quad (2.11)$$

From here on we may proceed in two ways.

2.1. Lagrangian algorithm

In the Lagrangian algorithm, we add the time derivative of the constraint $\Omega_1 = 0$ to the Lagrange equations of motion (2.7). This leads to the following enlarged set of equations (for a more detailed discussion see [21]),

$$\int d^2y \, W^{(1)}(x, y)\partial_0\xi(y) = K^{(1)}(x) \quad (2.12)$$

where $W^{(1)}(x, y)$ are now the elements of a *rectangular* matrix

$$W^{(1)}(x, y) = \begin{pmatrix} F^{(0)}(x, y) \\ M_1(x, y) \end{pmatrix} \quad (2.13)$$

with

$$M_{1\alpha}(x, y) = \frac{\partial\Omega_1(x)}{\partial\xi^\alpha(y)} = (2\vec{n}(x), \vec{0}, 0)\delta^2(x - y) \quad (2.14)$$

and

$$K^{(1)}(x) = \begin{pmatrix} K^{(0)}(x) \\ 0 \end{pmatrix}. \quad (2.15)$$

The matrix $W^{(1)}$ has two left zero modes, of which one just reproduces the previous constraint, while the other

$$v_y^{(1)}(x) = (\vec{0}, 2\vec{n}(x), 0, -1)\delta^2(x - y) \quad (2.16)$$

yields the new constraint

$$\Omega_2 = n^a\pi^a = 0. \quad (2.17)$$

---

[4] The superscript T on the vectors is implied.

We proceed in this fashion, and add the time derivative of this constraint to the previous equations of motion, which now read

$$\int d^2y \, W^{(2)}(x, y)\partial_0\xi(y) = K^{(2)}(x) \tag{2.18}$$

where $W^{(2)}(x, y)$ are now the elements of a *rectangular* matrix

$$W^{(2)}(x, y) = \begin{pmatrix} F^{(0)}(x, y) \\ M_1(x, y) \\ M_2(x, y) \end{pmatrix} \tag{2.19}$$

with

$$M_{2\alpha}(x, y) = \frac{\partial \Omega_2(x)}{\partial \xi^\alpha(y)} = (\vec{\pi}(x), \vec{n}(x), 0)\delta^2(x - y) \tag{2.20}$$

and

$$K^{(2)}(x) = \begin{pmatrix} K^{(0)}(x) \\ 0 \\ 0 \end{pmatrix}. \tag{2.21}$$

Now, $W^{(2)}$ is found to have three zero modes, of which two reproduce the previous constraints, while the zero mode

$$v_y^{(2)}(x) = (\vec{n}(x), -\vec{\pi}(x), 0, 0, 1)\delta^2(x - y) \tag{2.22}$$

leads to a further constraint,

$$\Omega_3 = 2n^0 n^a n^a + f\pi^a\pi^a + \frac{1}{f}n^a\vec{\nabla}^2 n^a = 0. \tag{2.23}$$

Extending the matrix $W^{(2)}$ to $W^{(3)}$ in the manner described above, we find that there are *no* new zero modes, and the iterative algorithm terminates at this point. We have thus four constraints, of which two constraints $\Omega_i = 0$ $(i = 1, 2)$ represent constraints among the variables n^a and π^a alone.

As one readily checks, the constraints are found to completely agree with the constraints generated by the conventional Dirac algorithm, departing either from the original second-order Lagrangian (2.1) or from the first-order Lagrangian (2.4).

2.2. Symplectic algorithm

In the symplectic algorithm of [19] we add the constraint Ω_1 to the canonical sector of the Lagrangian (2.4) in the form[5] $-\Omega_1\partial_0\rho$, thereby enlarging the symplectic phase space via the addition of a dynamical field ρ, which enforces the stability of the constraint Ω_1 in time. The once iterated first-level Lagrangian is then given as follows:

$$\mathcal{L}^{(1)} = \pi^a\partial_0 n^a - \Omega_1\partial_0\rho - \mathcal{H}^{(0)}. \tag{2.24}$$

In the spirit of [19] we absorb the Lagrange multiplier field n^0 in $\mathcal{H}^{(0)}$ into the new dynamical variable ρ by making the substitution $\partial_0\rho - n^0 \to \partial_0\rho$. This amounts to the replacement

$$\mathcal{L}^{(1)} \to \mathcal{L}^{(1)} = \pi^a\partial_0 n^a - \Omega_1\partial_0\rho - \mathcal{H}^{(1)}. \tag{2.25}$$

where $\mathcal{H}^{(0)}$ in equation (2.24) has been replaced by

$$\mathcal{H}^{(1)} = \frac{f}{2}\pi^a\pi^a + \frac{1}{2f}(\vec{\nabla}n^a)^2. \tag{2.26}$$

---

[5] The minus sign is included in order to align our results with those of the Lagrangian algorithm.

Note that at this stage the field n^0 has completely disappeared, and any information concerning it has been lost. In fact, the model that we are discussing at this point is the model considered in [27], and hence is no longer the O(3) NLSM in that sense.

The new equations of motion now read

$$\int d^2y\, F^{(1)}(x,y)\partial_0\xi^{(1)}(y) = K^{(1)}(x) = \begin{pmatrix} -\frac{1}{f}\vec{\nabla}^2 n^a(x) \\ f\pi^a(x) \\ 0 \end{pmatrix} \tag{2.27}$$

where

$$\xi^{(1)}(x) = (\vec{n}(x), \vec{\pi}(x), \rho(x)) \tag{2.28}$$

and the first-iterated symplectic two-form is now given by

$$F^{(1)}(x,y) = \begin{pmatrix} O & -I & -2\vec{n}(x) \\ I & O & \vec{0} \\ 2\vec{n}(x)^T & \vec{0}^T & 0 \end{pmatrix} \delta^2(x-y). \tag{2.29}$$

Note that the symplectic two-form matrix still has a zero mode $v_y^{(1)}(x) = (\vec{0}, 2\vec{n}(x), 1)\delta^2(x-y)$, which generates the constraint Ω_2 in the context of the symplectic formalism [19] as follows:

$$\int d^2y\, v_y^{(1)}(x) K^{(1)}(y) = 2f\Omega_2(x) = 0. \tag{2.30}$$

Following [19] we further enlarge the symplectic phase space to include the constraint Ω_2 via an additional dynamical field σ, as follows,

$$\mathcal{L}^{(2)} = \pi^a \partial_0 n^a - \Omega_1 \partial_0 \rho - \Omega_2 \partial_0 \sigma - \mathcal{H}^{(2)} \tag{2.31}$$

with $\mathcal{H}^{(2)} = \mathcal{H}^{(1)}$. The symplectic variables $\xi^{(2)}$ are now given by

$$\xi^{(2)}(x) = (n^a(x), \pi^a(x), \rho(x), \sigma(x)) \tag{2.32}$$

and the second-iterated symplectic two-form matrix is given by

$$F^{(2)}(x,y) = \begin{pmatrix} O & -I & -2\vec{n}(x) & -\vec{\pi}(x) \\ I & O & \vec{0} & -\vec{n}(x) \\ 2\vec{n}(x)^T & \vec{0}^T & 0 & 0 \\ \vec{\pi}(x)^T & \vec{n}(x)^T & 0 & 0 \end{pmatrix} \delta^2(x-y). \tag{2.33}$$

We easily show that this symplectic matrix is invertible, i.e. no zero modes exist. Hence the algorithm ends at this point.

Although the constraints $\Omega_i = 0\,(i = 1, 2)$ are identical with those generated by the Lagrangian algorithm, a complete analysis of the NLSM should have treated the Lagrange multiplier field n^0 as an additional degree of freedom, by not performing the shift in variable $\partial_0\rho - n^0 \to \partial_0\rho$. As shown in [21] the symplectic algorithm described above does not generate in this case the complete set of constraints as given by the Lagrangian algorithm, which rests on a solid foundation. The reason for this failure has been examined in [21] for the corresponding quantum mechanical model, as well as in general terms.

3. NLSM in the Hamilton–Jacobi scheme

In this section, we revisit the O(3) NLSM in the HJ scheme, for which the generalized HJ partial differential equations are given by[6]

$$\mathcal{H}_0' = p^0 + \mathcal{H}_0 = 0 \qquad \mathcal{H}_1' = \pi^0 + \mathcal{H}_1 = 0 \tag{3.1}$$

---

[6] This scheme was treated incompletely in [27]. With the identification $\pi^a(x) = \frac{\delta S}{\delta n^a(x)}$ and $p^0 = \partial_0 S$, these equations are the (incomplete set of) HJ partial differential equations for the Hamilton principal function S.

where $\mathcal{H}_0$ is a canonical Hamiltonian density $\mathcal{H}^{(0)}(n^0, n^a, \pi^a)$ in equation (2.3) and $\mathcal{H}'_1$ is the (only) primary constraint in the Dirac terminology [7] ($\mathcal{H}_1$ is actually zero for the case of the O(3) NLSM, as seen from equation (2.2)).

Following the generalized HJ scheme [23, 28], one obtains from equation (3.1)

$$\mathrm{d}q^{\underline{a}} = \frac{\partial \mathcal{H}'_{\underline{\alpha}}}{\partial p^{\underline{a}}} \, \mathrm{d}t_{\underline{\alpha}} \qquad \mathrm{d}p^{\underline{a}} = -\frac{\partial \mathcal{H}'_{\underline{\alpha}}}{\partial q^{\underline{a}}} \, \mathrm{d}t_{\underline{\alpha}} \tag{3.2}$$

where $q^{\underline{a}} = (t, n^0, n^a)$, $p^{\underline{a}} = (p^0, \pi^0, \pi^a)$ and $t_{\underline{\alpha}} = (t, n^0)$ with $\underline{\alpha} = (0, 1)$. Explicitly,

$$\partial_0 n^0 = \partial_0 n^0 \qquad\qquad \partial_0 n^a = f\pi^a$$
$$\partial_0 \pi^0 = n^a n^a - 1 \qquad \partial_0 \pi^a = \frac{1}{f}\vec{\nabla}^2 n^a + 2n^0 n^a. \tag{3.3}$$

Since the equation for n^0 is trivial, one cannot obtain any information about the variable n^0 at this level, and the set of equations is not integrable at this stage.

In order to fix this deficit of information, we supplement equations (3.1) with the generalized integrability conditions [23, 28]

$$\partial_0 \mathcal{H}'_{\bar{\alpha}} = \{\mathcal{H}'_{\bar{\alpha}}, \mathcal{H}'_0\} + \{\mathcal{H}'_{\bar{\alpha}}, \mathcal{H}'_{\beta}\}\partial_0 q^{\beta} = 0 \tag{3.4}$$

where, unlike in the usual case, the Poisson bracket is defined in terms of the extended index $\underline{a}$ corresponding to $q^{\underline{a}} = (t, n^0, n^a)$ as follows

$$\{A, B\} = \frac{\partial A}{\partial q^{\underline{a}}}\frac{\partial B}{\partial p^{\underline{a}}} - \frac{\partial B}{\partial q^{\underline{a}}}\frac{\partial A}{\partial p^{\underline{a}}}. \tag{3.5}$$

The index β in (3.4) labels the primary constraints, whose number is only one for the case in question ($\partial_0 q^{\beta} \to \partial_0 n^0$). The index $\bar{\alpha}$, on the other hand, labels the primary constraints as well as the secondary constraints which emerge *iteratively* from the integrability condition (3.4). Thus for the case in question we have three secondary constraints, $\mathcal{H}'_2 = 0, \mathcal{H}'_3 = 0$ and $\mathcal{H}'_4 = 0$, corresponding to equations (2.11), (2.17) and (2.23), respectively. They emerge iteratively from equation (3.4) as follows,

$$\partial_0 \mathcal{H}'_0 = -\mathcal{H}'_2 \partial_0 n^0 \qquad \partial_0 \mathcal{H}'_1 = \mathcal{H}'_2 \qquad \partial_0 \mathcal{H}'_2 = 2f\mathcal{H}'_3 \qquad \partial_0 \mathcal{H}'_3 = \mathcal{H}'_4$$

with

$$\mathcal{H}'_2 = n^a n^a - 1 \qquad \mathcal{H}'_3 = n^a \pi^a \qquad \mathcal{H}'_4 = 2n^0 n^a n^a + f\pi^a \pi^a + \frac{1}{f}n^a \vec{\nabla}^2 n^a. \tag{3.6}$$

From $\partial_0 \mathcal{H}'_4 = 0$ we have, using $\mathcal{H}'_2 = \mathcal{H}'_3 = 0$ as well as (3.3),

$$\partial_0 n^0 = (-\pi^a \vec{\nabla}^2 n^a + \vec{\nabla} n^a \cdot \vec{\nabla} \pi^a) \tag{3.7}$$

which replaces the identity $\partial_0 n^0 = \partial_0 n^0$ in equation (3.3), and thus completes the set of HJ equations to an integrable system.

Finally, note that equations (3.2) and (3.6) imply for the Hamilton principal function

$$\mathrm{d}S = \int \mathrm{d}^2 x \left(-\mathcal{H}_{\underline{\alpha}} + \pi^a \frac{\partial \mathcal{H}'_{\underline{\alpha}}}{\partial \pi^a}\right) \mathrm{d}t_{\underline{\alpha}}$$
$$= \mathrm{d}t \int \mathrm{d}^2 x \left(-\mathcal{H}_0 + \pi^0 \frac{\partial n^0}{\partial t} + \pi^a \frac{\partial n^a}{\partial t}\right). \tag{3.8}$$

Since we have now from equations (3.3) and (3.7) a complete set of equations of motion for n^a and n^0, equation (3.8) can be integrated in time to yield the standard action

$$S = \int \mathrm{d}^3 x \mathcal{L}^{(0)} \tag{3.9}$$

where $\mathcal{L}^{(0)}$ is the first-order Lagrangian (2.4). Note that it was only after taking account of the secondary constraints generated iteratively by the integrability condition (3.4), that one could construct the action (3.9) for the second-class system in question.

4. BFT Hamiltonian embedding

In this section, we reconsider the BFT quantization for the O(3) NLSM with the Lagrangian (2.1) and the canonical Hamiltonian (2.3).[7] The primary constraint reads

$$\Omega_0 = \pi^0 \approx 0 \tag{4.1}$$

and correspondingly we have for the total Hamiltonian

$$\mathcal{H}_T = \mathcal{H}_0 + v\Omega_0 \tag{4.2}$$

where $\mathcal{H}_0$ is the canonical Hamiltonian $\mathcal{H}^{(0)}$ in equation (2.3). The usual Dirac algorithm is readily shown to lead recursively to the constraints Ω_i ($i = 0, 1, 2, 3$) as follows[8]

$$\begin{aligned}
\{\Omega_0(x), \mathcal{H}_0(y)\} &= \Omega_1(x)\delta^2(x - y) \\
\{\Omega_1(x), \mathcal{H}_0(y)\} &= 2f\Omega_2(x)\delta^2(x - y) \\
\{\Omega_2(x), \mathcal{H}_0(y)\} &= \Omega_3(x)\delta^2(x - y).
\end{aligned} \tag{4.3}$$

The requirement of time independence of Ω_3 then finally fixes the Lagrange multiplier field in equation (4.2) to $v = 0$. Note that for the case of the motion of a particle on a sphere, the constraint $\Omega_3 = 0$ just fixes the 'string tension' of the central force governing its motion.

The complete system of constraints is thus second class. The subset $\Omega_1 = 0$ and $\Omega_2 = 0$ only links the fields n^a and π^a, while the remaining constraints $\Omega_0 = 0$ and $\Omega_3 = 0$ link these variables to the rest of the variables. We may thus 'gauge' the (n^a, π^a) sector in the style of BFT, while leaving a strong implementation of the constraints $\Omega_0 = 0$ and $\Omega_3 = 0$ to the end. This is possible since the corresponding Dirac brackets, constructed from the inverse of the matrix

$$\{\Omega_1(x), \Omega_2(y)\} = \begin{pmatrix} 0 & 2n^a(x)n^a(x) \\ -2n^a(x)n^a(x) & 0 \end{pmatrix} \delta^2(x - y) \tag{4.4}$$

reduce to ordinary Poisson brackets for functionals of n^a and π^a alone.

Following the BFT scheme [12–14], we systematically convert the second-class constraints $\Omega_i = 0$ ($i = 1, 2$) into first-class ones by introducing two canonically conjugate auxiliary fields (θ, π_θ) with Poisson brackets

$$\{\theta(x), \pi_\theta(y)\} = \delta^2(x - y). \tag{4.5}$$

The strongly involutive first-class constraints $\tilde{\Omega}_i$ are then constructed as a power series of the auxiliary fields [10],

$$\tilde{\Omega}_1 = \Omega_1 + 2\theta \qquad \tilde{\Omega}_2 = \Omega_2 - n^a n^a \pi_\theta \tag{4.6}$$

which satisfy the closed algebra $\{\tilde{\Omega}_i, \tilde{\Omega}_j\} = 0$.

We next construct the first-class fields $\tilde{\mathcal{F}} = (\tilde{n}^a, \tilde{\pi}^a)$, corresponding to the original fields defined by $\mathcal{F} = (n^a, \pi^a)$ in the extended phase space. They are obtained as a power series in the auxiliary fields (θ, π_θ) by demanding that they be in strong involution with the first-class constraints (4.6), that is $\{\tilde{\Omega}_i, \tilde{\mathcal{F}}\} = 0$. After some tedious algebra, we obtain for the first-class physical fields

$$\tilde{n}^a = n^a \left(\frac{n^a n^a + 2\theta}{n^a n^a}\right)^{1/2} \qquad \tilde{\pi}^a = (\pi^a - n^a \pi_\theta)\left(\frac{n^a n^a}{n^a n^a + 2\theta}\right)^{1/2}. \tag{4.7}$$

---

[7] In previous work [10] we have carried out the BFT embedding for the O(3) NLSM without explicitly including the geometrical constraint $n^a n^a - 1 = 0$ with the Lagrangian multiplier field n^0 in the starting Lagrangian.
[8] Note that these constraints correspond to $\mathcal{H}'_i = 0$ ($i = 1, 2, 3, 4$) in the HJ scheme, respectively.

They are found to satisfy the Poisson algebra

$$\{\tilde{n}^a(x), \tilde{\pi}^b(y)\} = \left(\delta^{ab} - \frac{\tilde{n}^a\tilde{n}^b}{\tilde{n}^c\tilde{n}^c}\right)\delta^2(x-y)$$

$$\{\tilde{\pi}^a(x), \tilde{\pi}^b(y)\} = \frac{1}{\tilde{n}^c\tilde{n}^c}(\tilde{n}^b\tilde{\pi}^a - \tilde{n}^a\tilde{\pi}^b)\delta^2(x-y). \tag{4.8}$$

To our knowledge, this is the first time that the first-class fields in the O(3) NLSM have been given in this compact form[9]. Note that in terms of the first-class fields (4.7) the constraints (4.6) take the natural form

$$\tilde{\Omega}_1 = \tilde{n}^a\tilde{n}^a - 1 \qquad \tilde{\Omega}_2 = \tilde{n}^a\tilde{\pi}^a. \tag{4.9}$$

Equation (4.9) illustrates that any functional $\mathcal{K}(\tilde{\mathcal{F}})$ of the first-class fields $\tilde{\mathcal{F}}$ is also first class. We correspondingly construct the first-class Hamiltonian in terms of the above first-class physical variables by making the replacements $n^a \to \tilde{n}^a, \pi^a \to \tilde{\pi}^a$ in the canonical Hamiltonian $\mathcal{H}_0$,

$$\tilde{\mathcal{H}}_0 = \frac{f}{2}\tilde{\pi}^a\tilde{\pi}^a + \frac{1}{2f}(\vec{\nabla}\tilde{n}^a)^2 - n^0(\tilde{n}^a\tilde{n}^a - 1). \tag{4.10}$$

Note that the first-class Hamiltonian (4.10) is manifestly strongly involutive with the first-class constraints, $\{\tilde{\Omega}_i, \tilde{\mathcal{H}}_0\} = 0$. This need not be so. Indeed, this Hamiltonian is not unique, as we may always add to it terms proportional to the first-class constraints without altering the dynamics of the first-class fields. Thus we could ask the gauged total Hamiltonian to generate the first-class constraints in an analogous way to equation (4.3)

$$\{\tilde{\Omega}_0(x), \tilde{\mathcal{H}}(y)\} = \tilde{\Omega}_1(x)\delta^2(x-y)$$
$$\{\tilde{\Omega}_1(x), \tilde{\mathcal{H}}(y)\} = 2f\tilde{\Omega}_2(x)\delta^2(x-y) \tag{4.11}$$
$$\{\tilde{\Omega}_2(x), \tilde{\mathcal{H}}(y)\} = \Omega_3(x)\delta^2(x-y).$$

This may be achieved via the replacement of (4.10) by

$$\tilde{\mathcal{H}} = \tilde{\mathcal{H}}_0 + f\pi_\theta\tilde{\Omega}_2 - \tfrac{1}{2}\ln(\tilde{n}^a\tilde{n}^a - 2\theta)\Omega_3. \tag{4.12}$$

Since the second-class constraint $\Omega_3 = 0$ should eventually be implemented strongly in our scheme, we shall set it equal to zero. As for $\tilde{\Omega}_2$, it is first class, so that it will not participate in the dynamics of gauge invariant functionals in the $(n^0, \pi^0, n^a, \pi^a, \theta, \pi_\theta)$ space. This allows us to set the last two terms in equation (4.12) equal to zero. Rewriting the Hamiltonian (4.10) in terms of the original fields in the (n^a, π^a) sector, and the auxiliary ones in the (θ, π_θ) sector, we have

$$\tilde{\mathcal{H}}_0 = \frac{f}{2}(\pi^a - n^a\pi_\theta)(\pi^a - n^a\pi_\theta)\frac{n^cn^c}{n^cn^c + 2\theta} + \frac{1}{2f}(\vec{\nabla}n^a)^2\frac{n^cn^c + 2\theta}{n^cn^c} - n^0(n^an^a - 1 + 2\theta) \tag{4.13}$$

where we have used the conformal map condition, $n^a\vec{\nabla}n^a = 0$, which states that the radial vector is perpendicular to the tangent on the S^2 sphere in the extended phase space of the O(3) NLSM [10].

In the framework of the BFV formalism [16, 17], we now construct the nilpotent BRST charge Q, the fermionic gauge fixing function Ψ and the BRST invariant minimal Hamiltonian

---

[9] Note that the first-class physical fields were obtained in terms of the infinite power series of the auxiliary fields (θ, π_θ) in the previous work [10].

H_m by introducing two canonical sets of ghost and anti-ghost fields, together with auxiliary fields $(\mathcal{C}^i, \bar{\mathcal{P}}_i)$, $(\mathcal{P}^i, \bar{\mathcal{C}}_i)$ and (N^i, B_i), $i = 1, 2$,

$$Q = \int d^2x(\mathcal{C}^i \tilde{\Omega}_i + \mathcal{P}^i B_i) \qquad \Psi = \int d^2x(\bar{\mathcal{C}}_i \chi^i + \bar{\mathcal{P}}_i N^i)$$

$$H_m = \int d^2x(\tilde{\mathcal{H}}_0 - 2f\mathcal{C}^1 \bar{\mathcal{P}}_2)$$

(4.14)

with the properties $Q^2 = \{Q, Q\} = 0$ and $\{\{\Psi, Q\}, Q\} = 0$. The nilpotent charge Q is the generator of the following infinitesimal transformations,

$$\delta_Q n^0 = 0 \qquad \delta_Q n^a = -\mathcal{C}^2 n^a \qquad \delta_Q \theta = \mathcal{C}^2 n^a n^a$$

$$\delta_Q \pi^0 = 0 \qquad \delta_Q \pi^a = 2\mathcal{C}^1 n^a + \mathcal{C}^2(\pi^a - 2n^a \pi_\theta) \qquad \delta_Q \pi_\theta = 2\mathcal{C}^1$$

$$\delta_Q \bar{\mathcal{C}}_i = B_i \qquad \delta_Q \mathcal{C}^i = 0 \qquad \delta_Q B_i = 0$$

$$\delta_Q \mathcal{P}^i = 0 \qquad \delta_Q \bar{\mathcal{P}}_i = \tilde{\Omega}_i \qquad \delta_Q N^i = -\mathcal{P}^i$$

(4.15)

which in turn imply $\{Q, H_m\} = 0$, that is, H_m in equation (4.14) is the BRST invariant minimal Hamiltonian.

After some algebra, we arrive at the effective quantum Lagrangian of the manifestly covariant form

$$\mathcal{L}_{eff} = \mathcal{L}^{(0)} + \mathcal{L}^{WZ} + \mathcal{L}^{ghost}$$

(4.16)

where $\mathcal{L}^{(0)}$ is given by equation (2.1) and

$$\mathcal{L}^{WZ} = \frac{1}{fn^c n^c}(\partial_\mu n^a)(\partial^\mu n^a)\theta - \frac{1}{2f(n^c n^c)^2}\partial_\mu \theta \partial^\mu \theta + 2n^0 \theta$$

$$\mathcal{L}^{ghost} = -\frac{1}{2f}(n^a n^a)^2(B + 2\bar{\mathcal{C}}\mathcal{C})^2 - \frac{1}{n^c n^c}\partial_\mu \theta \partial^\mu B + \partial_\mu \bar{\mathcal{C}} \partial^\mu \mathcal{C}.$$

(4.17)

This Lagrangian is invariant under the BRST transformation

$$\delta_\epsilon n^0 = 0 \qquad \delta_\epsilon n^a = \epsilon n^a \mathcal{C} \qquad \delta_\epsilon \theta = -\epsilon n^a n^a \mathcal{C}$$

$$\delta_\epsilon \bar{\mathcal{C}} = -\epsilon B \qquad \delta_\epsilon \mathcal{C} = 0 \qquad \delta_\epsilon B = 0$$

(4.18)

where ϵ is an infinitesimal Grassmann valued parameter. Note that the Wess–Zumino Lagrangian in (4.17) involves n^0, which originates from the geometrical constraint in the starting Lagrangian.

5. Conclusion

We have investigated the constraint structure of the O(3) NLSM, in the Lagrangian, symplectic, Hamilton–Jacobi (HJ) and Batalin–Fradkin–Tyutin (BFT) quantization schemes. In particular, we showed that the symplectic algorithm led to an incomplete constraint structure for this model. In fact, the missing constraint (2.23) was shown to play an essential role as a part of the integrability conditions in the HJ formulation, necessary to recover the action for this second-class system.

We further considered the gauge embedding of the NLSM following the BFT quantization scheme for constructing the first-class fields and first-class Hamiltonian including the Wess–Zumino terms. A similar study, based on the 'first-class field approach' was done in [27] in the framework of the HJ scheme. Even though the authors of [27] exploited the first-class Hamiltonian in a form similar to $\tilde{\mathcal{H}}_0$ in equation (4.10), their final Lagrangian is not covariant, in contrast to the covariant Lagrangian (4.16) involving the Wess–Zumino and

ghost fields. Moreover the 'integrability conditions' were not taken into account. The attempt to achieve such an embedding for the NLSM in the symplectic approach based on a biased 'educated guess' for the general Lorentz covariant form of the first-class gauged Lagrangian, as was done in [23] for the Proca and self-dual models, seems very problematic as seen from equation (4.17), which involves inverse powers of $(n^a n^a)$. We thus believe that for the case of the O(3) NLSM, the BFT embedding procedure is the most tractable when compared to the Lagrangian, symplectic and HJ schemes.

Acknowledgments

References

[1] Belavin A A and Polyakov A M 1975 *JETP Lett.* **22** 225
[2] Hong S T and Park Y J 2002 *Phys. Rep.* **358** 143 and references therein
[3] Wilczek F 1982 *Phys. Rev. Lett.* **48** 1144
    Wilczek F 1982 *Phys. Rev. Lett.* **49** 957
    Wilczek F and Zee A 1983 *Phys. Rev. Lett.* **51** 2250
    Wu Y S and Zee A 1984 *Phys. Lett.* B **147** 325
[4] Bowick M, Karabali D and Wijewardhana L C R 1986 *Nucl. Phys.* B **271** 417
    Kimura M, Kobayashi H and Tsutsui I 1998 *Nucl. Phys.* B **527** 624
[5] Semenoff G W 1988 *Phys. Rev. Lett.* **61** 517
    Panigrahi P K, Roy S and Scherer W 1988 *Phys. Rev. Lett.* **61** 2827
    Foerster A and Girotti H O 1989 *Phys. Lett.* B **230** 83
[6] Kovner A 1989 *Phys. Lett.* B **224** 229
    Bergeron M and Semenoff G W 1992 *Int. J. Mod. Phys.* A **7** 2417
    Banerjee N, Ghosh S and Banerjee R 1994 *Phys. Rev.* D **49** 1996
    Oh P and Pachos J 1998 *Phys. Lett.* B **444** 469
[7] Dirac P A M 1964 *Lectures on Quantum Mechanics* (New York: Yeshiba University Press)
[8] Hong S T, Kim Y W and Park Y J 1999 *Phys. Rev.* D **59** 114026
    Hong S T and Park Y J 2001 *Phys. Rev.* D **63** 054018
[9] Wong S M H 2002 *Preprint* hep-th/0207194
[10] Hong S T, Kim W T and Park Y J 1999 *Phys. Rev.* D **60** 125005
[11] Hong S T, Park Y J, Kubodera K and Myhrer F 2001 *Mod. Phys. Lett.* A **16** 1236
    Hong S T, Lee B H and Park Y J 2002 *Mod. Phys. Lett.* A **17** 103
    Chakraborty B 2002 *Mod. Phys. Lett.* A **17** 115
[12] Batalin I A and Fradkin E S 1986 *Phys. Lett.* B **180** 157
    Batalin I A and Fradkin E S 1987 *Nucl. Phys.* B **279** 514
    Batalin I A and Tyutin I V 1991 *Int. J. Mod. Phys.* A **6** 3255
[13] Banerjee R 1993 *Phys. Rev.* D **48** R5467
    Kim W T and Park Y J 1994 *Phys. Lett.* B **336** 376
[14] Kim Y W, Park Y J and Rothe K D 1998 *J. Phys. G: Nucl. Part. Phys.* **24** 953
    Kim Y W and Rothe K D 1998 *Nucl. Phys.* B **510** 511
    Park M I and Park Y J 1998 *Int. J. Mod. Phys.* A **13** 2179
    Harikumar E 2001 *Nucl. Phys.* B **618** 570
[15] Becci C, Rouet A and Stora R 1976 *Ann. Phys., NY* **98** 287
    Tyutin I V 1975 Unpublished
[16] Fradkin E S and Vilkovisky G A 1975 *Phys. Lett.* B **55** 224
    Henneaux M 1985 *Phys. Rep.* **126** 1
[17] Fujiwara T, Igarashi Y and Kubo J 1990 *Nucl. Phys.* B **341** 695
    Kim Y W, Kim S K, Kim W T, Park Y J, Kim K Y and Kim Y 1992 *Phys. Rev.* D **46** 4574

[18] Faddeev L and Jackiw R 1985 *Phys. Rev. Lett.* **60** 1692
    Jackiw R 1995 *Diverse Topics in Theoretical and Mathematical Physics* (Singapore: World Scientific)
[19] Barcelos-Neto J and Wotzasek C 1992 *Mod. Phys. Lett.* A **7** 1172
    Barcelos-Neto J and Wotzasek C 1992 *Int. J. Mod. Phys.* A **7** 4981
[20] Kim S K, Kim Y W, Park Y J, Kim Y, Kim C H and Kim W T 1995 *J. Korean Phys. Soc.* **28** 128
[21] Rothe H J and Rothe K D 2002 *Preprint* hep-th/0208106
[22] Mendes A C R, Neto J A, Oliveira W, Neves C and Rodrigues D C 2001 *Preprint* hep-th/0109089
[23] Hong S T, Kim Y W, Park Y J and Rothe K D 2002 *Mod. Phys. Lett.* A **17** 435
    Hong S T, Kim Y W, Park Y J and Rothe K D 2002 *J. Phys. A: Math. Gen.* **35** 7461
[24] Carathéodory C 1967 *Calculus of Variations and Partial Differential Equations of First Order, Part II*
    (San Francisco, CA: Holden-Day)
    Kastrup H A 1983 *Phys. Rep.* **101** 1
[25] Güler Y 1989 *J. Math. Phys.* **30** 785
[26] Pimentel B M, Teixeira R G and Tomazelli J L 1998 *Ann. Phys.* B **267** 75
[27] Baleanu D and Güler Y 2001 *Mod. Phys. Lett.* A **16** 873
[28] Hong S T, Kim W T, Kim Y W and Park Y J 2001 *Preprint* quant-ph/0109032
[29] Dominici D, Gomis J, Longhi G and Pons J M 1984 *J. Math. Phys.* **25** 2439